The Lift Force Due to von Kármán's Vortex Wake

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Equations are developed which permit the estimation of the oscillating lift forces on bluff cylindrical bodies. The coefficient of lift is seen to be a function of the Strouhal number, the ratio of the lateral to longitudinal vortex spacing of the wake vortices and the ratio of the lateral vortex spacing to the width of the body. Introduction of certain simplifying assumptions permit the determination of the lift coefficient of any bluff cylindrical body, provided the Strouhal number and the drag coefficient of that body at the Reynolds number of interest are known. The equations also predict an increase in the lift coefficient if the cylinder vibrates with a frequency less than the Strouhal frequency of the stationary cylinder and a decrease in lift coefficient if the cylinder vibrates at a higher frequency.

Nomenclature

d = width of body
D = drag force acting on body per unit length
h = lateral vortex spacing
l = longitudinal vortex spacing
L = lift force acting on body per unit length
M_x = momentum of fluid within control volume in the x-direction
M_y = momentum of fluid within control volume in the y-direction
p = static pressure
$Re\{A\}$ = real part of A
R = Reynolds number
S = Strouhal number
t = time
u = velocity component in x-direction
$u'(x,y)$ = velocity in x-direction due to vortex street
u_s = translational velocity of vortex street with respect to the fluid
U = velocity of vortex shedding body with respect to the fluid
v = velocity component of y-direction
$v'(x,y)$ = velocity in y-direction due to vortex street
w = complex potential
x,y = coordinates in the complex plane $z = x + iy$
Γ = circulation of one vortex
Γ_t = total circulation within the control volume
ρ = density of fluid
φ = velocity potential
ψ = stream function

Introduction

WHEN a bluff cylindrical body is exposed to an uniform cross flow of a viscous fluid a fluctuating lift force acts on the body in addition to the nearly steady drag force. In his classic paper von Kármán[1] succeeded in estimating the steady drag force acting on a cylindrical body by employing potential flow considerations. This paper extends von Kármán's analysis by deriving an expression for the fluctuating lift force acting on the body due to the formation of the Kármán vortex street. In the initial part of the analysis the momentum approach as described by von Kármán and J. M. Burgers in Durand[2] is closely followed. The final part of the analysis employs the method of relating the dimensionless longitudinal vortex spacing l/d to the Strouhal number S and the coefficient of drag C_D as proposed by Sallet.[3] The basic assumptions upon which the momentum analysis given, is based on the following points. 1) The cylindrical body is of uniform cross section and has a large enough slenderness ratio so that a two-dimensional analysis is justified. 2) A regular Kármán vortex street is generated by the uniform motion of the bluff cylindrical body. 3) The momentum of any vortex pair within the real wake is the same as the momentum of the equivalent vortex pair of the potential flow model. 4) The drag force experienced by the cylindrical body is only due to the generated vortex street and not due to the viscous forces acting directly on the body. 5) The complex potential of the infinite Kármán vortex street does not significantly differ from the complex potential of the semi-infinite vortex street which represents the wake of the cylindrical body.

The technical significance of the lift forces due to von Kármán's vortex wake was described by Den Hartog.[4] Earlier estimates of the lift forces considered only cylinders having a circular cross section, e.g., Ruedy,[5] Steinman,[6] Schwabe,[7] and Chen.[8] The present investigation is valid for any bluff body in cross flow, as long as the stated assumptions are satisfied.

Momentum Analysis

Let the complex potential of the flowfield be denoted by

$$w(z) = \varphi(x, y) + i\psi(x, y) \tag{1}$$

The complex potential of a single line vortex at $z_o = x_o + iy_0$ is

$$w(z) = -i\frac{\Gamma}{2\pi}\ln(z - z_o) \tag{2}$$

where the circulation Γ is positive when the fluid motion due to the vortex is counterclockwise. For the infinite double row, staggered vortex street shown in Fig. 1, the complex potential is found to be

$$w(z) = i\frac{\Gamma}{2\pi}\ln\frac{\sin\pi/l(z - z_o)}{\sin\pi/l(z + z_o)} \tag{3}$$

where

$$z_o = \pm\, l/4 + i(h/2) \tag{4}$$

The positive sign in front of the term $l/4$ is valid if the imaginary axis is located as shown by the fully drawn iy-axis in Fig. 1, while the negative sign is valid for the dotted iy-axis. From the complex velocity of the vortex sys-

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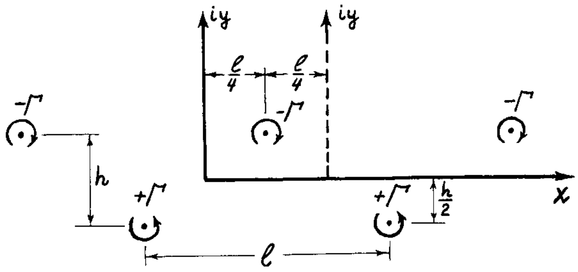

Fig. 1 Vortex street.

tem,

$$\frac{dw}{dz} = u'(x,y) - iv'(x,y) = i\frac{\Gamma}{2l}\left[\cot\frac{\pi}{l}(z - z_o) - \cot\frac{\pi}{l}(z + z_o)\right] \quad (5)$$

it is found that the total vortex street propagates with a velocity

$$u_s = \frac{\Gamma}{2l}\tanh\left(\frac{\pi h}{l}\right) \quad (6)$$

to the left. The blunt cylinder which causes the vortex street moves with a uniform velocity U along the negative real axis also to the left. In order to make the vortex system stationary a velocity u_s from left to right and parallel to the real axis is superposed upon the moving vortex system. The boundaries of the stationary control volume may now be selected as shown in Fig. 2. The boundary $\overline{DA}$ may be drawn as shown in Fig. 2a and 2c so that $z_o = +l/4 + ih/2$ or $\overline{DA}$ may be drawn as shown in Fig. 2b and 2d, so that $z_o = -l/4 + ih/2$. The cylinder moves with the velocity $(U - u_s)$ to the left and stays at all times within the control volume. The vortex system is at rest. The velocity at $x = -\infty$ and $y = \pm\infty$ is u_s. The momentum theorem can now be stated in the form

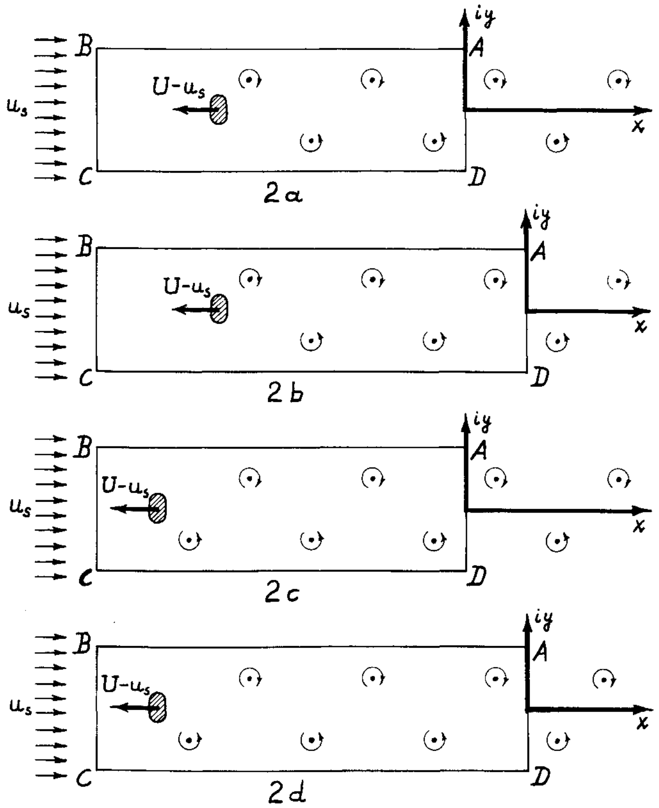

Fig. 2 Control volume boundaries.

$$L = -\frac{dM_y}{dt} + \int_C^D p dx - \int_B^A p dx + \int_C^D \rho v^2 dx - \int_B^A \rho v^2 dx + \int_C^B \rho uv dy - \int_D^A \rho uv dy \quad (7)$$

where the velocity components u and v are

$$\left.\begin{aligned} u &= u' + u_s \\ v &= v' \end{aligned}\right\} \quad (8)$$

The first term on the right hand in Eq. (7) is the time rate of change of the momentum of the fluid in the y-direction within the control volume; the second and third give the net force in the y-direction on the control volume; the last four terms give the net momentum inflow in the y-direction. Setting

$$p = \text{const} - \frac{\rho}{2}(u^2 + v^2) - \rho\frac{\partial\varphi}{\partial t} \quad (9)$$

Equation (7) yields

$$L = -\frac{dM_y}{dt} - \rho\frac{d}{dt}\int_C^D \varphi dx + \rho\frac{d}{dt}\int_B^A \varphi dx - \frac{\rho}{2}\int_C^D (u^2 - v^2)dx + \frac{\rho}{2}\int_B^A (u^2 - v^2)dx + \rho\int_B^C uvdy - \rho\int_D^A uvdy$$

By introducing the notation given by Eq. (8), the preceding expression becomes

$$L = -\frac{dM_y}{dt} - \rho\frac{d}{dt}\int_C^D \varphi dx + \rho\frac{d}{dt}\int_B^A \varphi dx - \frac{\rho}{2}\int_C^D (u'^2 + 2u'u_s + u_s^2 - v'^2)dx + \frac{\rho}{2}\int_B^A (u'^2 + 2u'u_s + u_s^2 - v'^2)dx + \rho\int_C^B (u'v' + u_s v')dy - \rho\int_D^A (u'v' + u_s v')dy$$

The terms with the expression u_s^2 cancel and all terms of the last 4 integrals which do not have the factor u_s vanish at three sides of the control surface, namely sides AB, BC, and CD. The equation for the lift therefore is

$$L = -\frac{dM_y}{dt} - \rho\frac{d}{dt}\int_C^D \varphi dx + \rho\frac{d}{dt}\int_B^A \varphi dx - \rho u_s\left[\int_C^D u'dx + \int_D^A v'dy + \int_A^B u'dx + \int_B^C v'dy\right] - \rho\int_D^A u'v'dy \quad (10)$$

It is seen that the fourth term on the right-hand side of Eq. (10) is the product of the density ρ, the velocity u_s and the total circulation Γ_t within the control volume. In von Kármán's analysis of the drag force D, i.e., the force in the x-direction, such a circulation term does not appear:

$$D = -\frac{dM_x}{dt} - \rho\frac{d}{dt}\int_C^B \varphi dy + \rho\frac{d}{dt}\int_D^A \varphi dy + \rho u_s\left[\int_C^B u'dx - \int_D^A u'dy + \int_C^D v'dx - \int_B^A v'dx\right] - \frac{\rho}{2}\int_D^A (u'^2 - v'^2)dy \quad (11)$$

The fourth term on the right-hand side of Eq. (11) reduces to zero due to continuity.

Since

$$\rho \int_D^A u'v'dy = Re\{\frac{\rho}{2}\int_D^A (u' - iv')^2 dz\} \quad (12)$$

Equation (10) becomes

$$L = -\frac{dM_y}{dt} + \rho\frac{d}{dt}[\int_B^A \varphi dx - \int_C^D \varphi dx] - \rho u_s \Gamma_t - Re\{\frac{\rho}{2}\int_D^A (u' - iv')^2 dz\} \quad (13)$$

For the evaluation of the integrals in Eq. (13) the control volume boundaries $\overline{CD}$ and $\overline{AB}$ are moved far away from the real axis. From Eq. (3) one obtains†

as $y \to +\infty$ and $x = 0$

$$(\varphi + i\psi) \to (-\frac{\Gamma}{4} - i\frac{\Gamma h}{2l}) \text{ if } z_o = +\frac{l}{4} + i\frac{h}{2} \quad (14)$$

$$(\varphi + i\psi) \to (+\frac{\Gamma}{4} - i\frac{\Gamma h}{2l}) \text{ if } z_o = -\frac{l}{4} + i\frac{h}{2} \quad (15)$$

as $y \to +\infty$ and $x = 0$

$$(\varphi + i\psi) \to (+\frac{\Gamma}{4} + i\frac{\Gamma h}{2l}) \text{ if } z_o = +\frac{l}{4} + i\frac{h}{2} \quad (16)$$

$$(\varphi + i\psi) \to (-\frac{\Gamma}{4} + i\frac{\Gamma h}{2l}) \text{ if } z_o = -\frac{l}{4} + i\frac{h}{2} \quad (17)$$

The second term on the right side of Eq. (13) is evaluated by comparing the velocity potential along the boundaries $\overline{BA}$ and $\overline{CD}$ of the vortex configuration shown in Fig. 2a or 2b with the potential along the same boundaries one half vortex shedding period later, as shown in Fig. 2c or 2d, respectively. It is seen that the time rate of change of the integral of the velocity potential along $\overline{BA}$ and $\overline{CD}$ at $y = \pm\infty$ vanishes. The last integral of Eq. (13) becomes

$$\int_D^A (u' - iv')^2 dz = \int_{z=-i\infty}^{z=+i\infty} (\frac{dw}{dz})^2 dz \quad (18)$$

Integrating gives‡

$$\int_{z=-i\infty}^{z=+i\infty} (\frac{dw}{dz})^2 dz = +\frac{\Gamma^2}{4\pi l}[\cot\frac{\pi}{l}(z - z_o) + \cot\frac{\pi}{l}(z + z_o) + 2\cot\frac{2\pi z_o}{l}\ln\frac{\sin\pi/l(z - z_o)}{\sin\pi/l(z + z_o)}]_{z=-i\infty}^{z=+i\infty} \quad (19)$$

For the evaluation of Eq. (19) the following expressions are derived:

$$\cot\frac{2z_o\pi}{l} = -i\tanh(\pi h/l) \quad (20)$$

as

$$y \to +\infty: \cot\pi/l(z - z_o) \to -i \quad (21)$$

$$\cot\pi/l(z + z_o) \to -i \quad (22)$$

$$\ln\frac{\sin\pi/l(z - z_o)}{\sin\pi/l(z + z_o)} \to \ln[+i\exp(-\frac{\pi h}{l})] \quad (23)$$

$$\text{if } z_o = +\frac{l}{4} + i\frac{h}{2}$$

$$\ln\frac{\sin\pi/l(z - z_o)}{\sin\pi/l(z + z_o)} \to \ln[-i\exp(-\frac{\pi h}{l})] \quad (24)$$

$$\text{if } z_o = -\frac{l}{4} + i\frac{h}{2}$$

as

$$y \to -\infty: \cot\pi/l(z - z_o) \to +i \quad (25)$$

$$\cot\pi/l(z + z_o) \to +i \quad (26)$$

$$\ln\frac{\sin\pi/l(z - z_o)}{\sin\pi/l(z + z_o)} \to \ln[-i\exp(\frac{\pi h}{l})] \quad (27)$$

$$\text{if } z_o = +\frac{l}{4} + i\frac{h}{2}$$

$$\ln\frac{\sin\pi/l(z - z_o)}{\sin\pi/l(z + z_o)} \to \ln[+i\exp(\frac{\pi h}{l})] \quad (28)$$

$$\text{if } z_o = -\frac{l}{4} + i\frac{h}{2}$$

The evaluation of Eq. (19) now becomes§

$$\int_{z=-i\infty}^{z=+i\infty} (\frac{dw}{dz})^2 dz = -\frac{\Gamma^2}{\pi l}[i + (\frac{\pi}{2} - \frac{\pi h}{l}i)\tanh(\frac{\pi h}{l})] \quad (29)$$

$$\text{if } z_o = +\frac{l}{4} + i\frac{h}{2}$$

and

$$\int_{z=-i\infty}^{z=+i\infty} (\frac{dw}{dz})^2 dz = -\frac{\Gamma^2}{\pi l}[i + (-\frac{\pi}{2} - \frac{\pi h}{l}i)\tanh(\frac{\pi h}{l})] \quad (30)$$

$$\text{if } z_o = -\frac{l}{4} + i\frac{h}{2}$$

Equation (13) now becomes

$$L = -\frac{dM_y}{dt} - \rho u_s\Gamma_t + \frac{1}{2}\rho u_s\Gamma \quad (31)$$

$$\text{if } z_o = +\frac{l}{4} + i\frac{h}{2}$$

and

$$L = -\frac{dM_y}{dt} - \rho u_s\Gamma_t - \frac{1}{2}\rho u_s\Gamma \quad (32)$$

$$\text{if } z_o = -\frac{l}{4} + i\frac{h}{2}$$

The result of this momentum consideration should clearly not depend upon the choice of the control volume. Consider Fig. 2a. The time rate of change of momentum of the fluid is positive. The number of vortices in the control volume is even, which results in zero total circulation, $\Gamma_t = 0$. Equation (31) therefore becomes

$$L = -\frac{dM_y}{dt} + \frac{1}{2}\rho u_s\Gamma \quad (33)$$

Shifting the right vertical control surface a distance $l/2$ to the right (Fig. 2b) or to the left will result in an uneven number of vortices within the control volume; there is one extra vortex of clockwise rotation. Therefore $\Gamma_t = -\Gamma$ and Eq. (32) yields Eq. (33). Half a vortex shedding period later (Fig. 2c) the time rate of change of momentum of the fluid is negative. The number of vortices in the control volume is odd; there is one extra vortex of counterclockwise rotation. Therefore Γ_t equals $+\Gamma$ and Eq. (31) yields

$$L = +\frac{dM_y}{dt} - \frac{1}{2}\rho u_s\Gamma \quad (34)$$

Shifting the right vertical control surface a distance $l/2$ to the right (Fig. 2d) or to the left will yield again the result given in Eq. (34) after employing Eq. (32). The equation for the lift may, therefore, be written as

$$L = \pm[\frac{dM_y}{dt} - \frac{1}{2}\rho u_s\Gamma] \quad (35)$$

† Note the printing errors occuring in the equivalent expressions by von Kármán[1] pp. 56, center of right column.

‡ Note the printing error in Durand,[2] pp. 347, Eq. (7.4).

§ Note the two printing errors in Durand,[2] pp. 347 in the equation which follows Eq. (7.4).

The period of the vortex street can be expressed in terms of the longitudinal vortex spacing l and the velocities of the cylinder and the vortex street by the relation

$$\Delta t = l/(U - u_s) \quad (36)$$

Every vortex in the vortex street can be thought of as part of two vortex pairs; each vortex forming such a vortex pair has the circulation $\Gamma/2$. The total positive or the total negative momentum in the y-direction created during one period Δt is now seen to be

$$M_y = (\rho/4)l\Gamma \quad (37)$$

The net sum of the positive and the negative momentum produced during one period is zero. Equation (35) can now be written as

$$L = \pm\frac{1}{4}\rho\Gamma[U - 3u_s] \quad (38)$$

The latter equation further reduces to

$$S^2(\frac{l}{d})^3 + 0.529S(\frac{l}{d})^2 - 1.529\frac{l}{d} + 1.593C_D = 0 \quad (46)$$

From momentum considerations it is plausible to assume that the lift varies as a function of time in a similar functional manner as the separatrix of the stream function varies as a function of distance along the real axis. Setting $z_o = +l/4 + i(h/2)$ the imaginary part of Eq. (3) becomes

$$\psi = -\frac{\Gamma}{4\pi}\ln\frac{\sin^2[(x+\frac{l}{4})\frac{\pi}{l}]\cosh^2[(y+\frac{h}{2})\frac{\pi}{l}] + \cos^2[(x+\frac{l}{4})\frac{\pi}{l}]\sinh^2[(y+\frac{h}{2})\frac{\pi}{l}]}{\sin^2[(x-\frac{l}{4})\frac{\pi}{l}]\cosh^2[(y-\frac{h}{2})\frac{\pi}{l}] + \cos^2[(x-\frac{l}{4})\frac{\pi}{l}]\sinh^2[(y-\frac{h}{2})\frac{\pi}{l}]} \quad (47)$$

Setting $\psi = 0$ and making use of Eq. (43) yields

$$\sin(\frac{2\pi x}{l}) + \sinh(\frac{2\pi y}{l}) = 0 \quad (48)$$

Since generally $y/d < 1$ and since $l/d > \pi$

$$y \approx (\text{const})\sin\frac{2\pi x}{l} \quad (49)$$

and, with the aforementioned assumption,

$$C_L(t) = C_L\sin\frac{2\pi US}{d}t \quad (50)$$

For a given Strouhal number S and a given coefficient of drag C_D the longitudinal vortex spacing distance l can be calculated in terms of the width of the body by means of Eq. (46). Equation (44) can then be used to determine the coefficient of lift C_L. For example, Schwabe[7] experimentally determined the lift coefficient for a circular cylinder at a Reynolds number of 735 to be $C_L = 0.447$. He also measured the drag coefficient and found $C_D = 1.09$. At the above Reynolds number the Strouhal number of a circular cylinder equals 0.210.[9] Substituting the above values of C_D and S into Eq. (46) yields the dimensionless longitudinal vortex spacing $l/d = 3.84$. Substituting this value and the above Strouhal number into Eq. (44) yields a coefficient of lift of $C_L = 0.441$. At a Reynolds number of 2×10^5 Fung[11] experimentally determined the coefficient of lift of circular cylinders to lie between 0.30 and 0.39 while at a Reynolds number of 6×10^5 he found the lift coefficients to have values between 0.18 and 0.20. Using the values $S = 0.21$ and $C_D = 1.20$ at $R = 2 \times 10^5$ and $S = 0.29$ and $C_D = 0.30$ at $R = 0.30$ (Roshko[11]) yields coefficients of lift of 0.382 and 0.225 for the lower and higher Reynolds number respectively.

The preceding examples show acceptable agreement between the experimentally obtained coefficient of lift and by Eq. (44) predicted coefficient of lift in the subcritical as well as in the supercritical Reynolds number range. The rather close agreement of the predicted value with the experimental value given by Schwabe[7] may be misleading. It should be pointed out, that Eq. (38) and, therefore, Eqs. (41, 42, and 44) are based on the total momentum increase during one half of a vortex shedding period, while the experimental values quoted above are maximum values occuring instantaneously. However, the method of splitting each vortex into two vortices having half the circulation of the original vortex and considering each of these vortices as making up a vortex pair, one with a similar half strength vortex of the preceeding vortex and the other with a half strength vortex of the succeding vortex of the opposite row results in an estimation of the maximum rather than the average momentum increase.

It is of interest that Eq. (41) and consequently also Eq. (44) predict a change in lift force when the cylindrical body vibrates, as was stipulated by Den Hartog[4] and experimentally investigated by Bishop and Hassan.[12] The Strouhal number S of the stationary cylinder is now re-

Evaluation and Discussion

It is of interest to compare the lift given by Eq. (38) with experimentally obtained values. Neither the circulation Γ nor the vortex street velocity u_s is normally measured during lift and drag measurements. However, the circulation and the vortex street velocity can be expressed in terms of the coefficient of drag and the Strouhal number. For blunt cylinders such as plates wedges and circular cylinders the drag coefficients and the vortex shedding frequency as functions of Reynolds number are readily available in the literature.

Equating the frequency of vortex shedding of the above potential flow model to the vortex shedding frequency which is experimentally observed yields

$$\frac{U - u_s}{U} = S\frac{l}{d} \quad (39)$$

Defining the coefficient of lift in the customary manner

$$C_L = 2L/\rho U^2 d \quad (40)$$

yields

$$C_L = \frac{l}{d}(1 - \frac{Sl}{d})(3\frac{Sl}{d} - 2)\coth(\frac{\pi h}{l}) \quad (41)$$

A similar development[3] for the coefficient of drag yields

$$C_D = \frac{4}{d}[h(1 - \frac{Sl}{d})(\frac{2Sl}{d} - 1)\coth(\frac{\pi h}{l}) + \frac{l}{\pi}(1 - \frac{Sl}{d})^2\coth^2(\frac{\pi h}{l})] \quad (42)$$

If von Kármán's stability criterion, namely

$$\cosh(\frac{\pi h}{l}) = (2)^{1/2}$$

or (43)

$$\frac{h}{l} = 0.280549$$

is assumed to give the true vortex spacing ratio, Eqs. (41 and 42) can be reduced to

$$C_L = (2)^{1/2}\frac{l}{d}(1 - \frac{Sl}{d})(3\frac{Sl}{d} - 2) \quad (44)$$

and

$$C_D = 4(2)^{1/2}\frac{l}{d}[1 - \frac{Sl}{d}][\frac{h}{l} - 2(1 - \frac{Sl}{d})(\frac{h}{l} - \frac{1}{\pi\sqrt{2}})] \quad (45)$$

placed by the value $k \cdot S$, where k lies between 0.8 and 1.2, see Ref. 4. For this range of values of k the vortex shedding takes place at the frequency with which the cylinder vibrates. For small amplitude vibrations the coefficient of drag may be assumed to be the same as for a steady cylinder. With this assumption the longitudinal vortex spacing and the coefficient of lift may then be calculated with Eqs. (42 or 46, and 41 or 44), respectively. For example, at a R of 6000 the coefficient of drag for a circular cylinder is 1 and the Strouhal number is 0.205. For values of $k = 0.9$, 1 and 1.05 Eq. (46) yields the values of l/d of 4.60, 4.03 and 3.76, respectively. Equation (44) now predicts coefficients of lift for $k = 0.9$, 1 and 1.05 of $C_L =$ 0.538, 0.473 and 0.436, respectively. The ratios of C_L at $k = 0.9$ to C_L at $k = 1$ and C_L at $k = 1.05$ to C_L at $k = 1$ therefore become 1.12 and 0.93, respectively. Experiments by Bishop and Hassan[12] at approximately the same Reynolds number show these ratios to be 1.21 and 0.80 (extrapolated). The experimental results were found when the ratio of the amplitude of the vibration to the diameter of the cylinder was 0.3. The drag force acting on a cylinder which is vibrating transversely to the flow may be estimated by substituting the projected area of the stationary cylinder with the area based on the extremal positions of the vibrating cylinder. An equivalent estimate of the drag in the same proportion as the effective area increases. Since Eq. (46) is dependent upon the drag force the effect of the large vibrational amplitude may be taken into account by using for the above given numerical example the fictious coefficient of drag of 1.30 (1 + 0.30) in the prediction equations. The predicted ratios of C_L at $k = 0.9$ to C_L at $k = 1$ and C_L at $k = 1.05$ to C_L at $k = 1$ become 1.43 and 0.72, respectively, when C_D takes the value of 1.3. The predicted increase in longitudinal vortex spacing when the cylinder is vibrating at a frequency which is lower that the vortex shedding frequency of a stationary cylinder and a decrease in the vortex spacing when the cylinder is vibrating at a higher frequency was qualitatively observed by Koopmann.[13]

In all of the above numerical evaluations for the lift coefficient and for the longitudinal vortex spacing equations were used which included von Kármán's value for h/l. In view of the strong influence which the vortex spacing has upon the predicted values of the coefficient of lift this ratio should also be considered a variable and experimently determined as a function of the Reynolds number. The fact that the potential vortex street becomes more unstable for values of h/l other than 0.280549 is of secondary importance, since the purpose of the potential flow model is to predict the lift forces due to a hypothetical vortex street which is so constructed that it correctly predicts the easily measured drag forces, and not to predict the flow pattern of the real wake. For this reason valid predictions for the lift will be obtained even though assumptions 2 and 3 may not be strictly justified. The experimentally observed Strouhal number and the experimentally obtained coefficient of drag are introduced into the equations describing a hypothetical vortex street. As a basis for this hypothetical model the Kármán vortex street was chosen. From Eqs. (3) and (4) it is seen that the unknowns in this potential flow model are the circulation Γ and the vortex spacing parameters h and l. By introducing the experimentally found values of the Strouhal number and coefficient of drag the unknowns h, l, and Γ are eliminated. The resulting vortex street model which is now so stipulated that it correctly predicts the drag is then used in the determination of the lift.

The time rate of change of momentum of the fluid was evaluated by the increase in momentum due to one vortex pair, i.e., the product of the density of the fluid, the circulation of the vortex and the distance between the two vortices making up the vortex pair. The result for the momentum increase obtained in this manner is identical to the result obtained by the integration of the stream function as was shown by von Kármán.[14] The evaluation of the momentum rate of change cannot be improved by taking a semi-infinite vortex street as the flow model of the wake. Such a semi-infinite wake model was first used by Synge[15] for the calculation of the drag acting on a cylindrical body. Synge's drag equation is identical to von Kármán's (Refs. 1 and 14) drag equation; the intermediate results of the evaluation of the momentum change are also the same. While Synge's semi-infinite vortex street model will make assumption 5 appear to be a more plausible assumption, the use of a semi-infinite vortex street instead of von Kármán's vortex street will make the equations mathematically somewhat more complex.

References

[1]von Kármán, Th. and Rubach, H., "Über den Mechanismus des Flüssigkeits und Luftwiderstandes," *Physikalische Zeitschrift*, No. 2, Jan. 1912, pp. 49–59.

[2]Durand, W. F., *Aerodynamic Theory*, Vol. II, Julius Springer Verlag, Berlin, 1935.

[3]Sallet, D. W., "On the Spacing of Kâmân Vortices," *Transactions of the ASME: Journal of Applied Mechanics*, June 1969, pp. 370–372.

[4]Den Hartog, J. P., "Recent Technical Manifestations of von Kármán's Vortex Wake," *Proceedings of the National Academy of Sciences*, Vol. 40, No. 3 1954, pp. 155–157.

[5]Ruedy, R., "Vibrations of Power Lines in a Steady Wind," *Canadian Journal of Research*, Vol. 13, Sec. A, No. 4, Oct. 1935, pp. 82–92.

[6]Steinman, D. B., "Problems of Aerodynamic and Hydrodynamic Stability," *Proceedings of the 3rd Hydraulics Conference*, Iowa Univ., Bulletin 31, 1947, pp. 136–163.

[7]Schwabe, M., "Über Druckermittlung in der nichtstationären ebenen Strömung," *Ingenieur-Archiv*, Vol. 6, Feb. 1935, pp. 34–50.

[8]Chen, Y. N., "Fluctuating Lift Forces of the Kármán Vortex Street on Single Circular Cylinders and in Tube Bundles," ASME Paper 71-Vibr. 12, Dec. 1971.

[9]Roshko, A., "On the Development of Turbulent Wakes From Vortex Streets." Rept. 1191, 1954, NACA.

[10]Fung, Y. C., "Fluctuating Lift and Drag Acting on a Cylinder in a Flow at Supercritical Reynolds Numbers," *Journal of Aerospace Science*, Vol. 27, No. 11, Nov. 1960, pp. 801–814.

[11]Roshko, A., "Experiments on the Flow Past a Circular Cylinder at Very High Reynolds Number," *Journal of Fluid Mechanics*, Vol. 10, Pt. 3, May 1961, pp. 345–356.

[12]Bishop, R. E. D. and Hassan, A. Y., "The lift and drag forces on a circular cylinder oscillating in a flowing fluid," *Proceedings of the Royal Society London*, Ser. A, Vol. 277, 1964, pp. 51–75.

[13]Koopmann, G. H., "The vortex wakes of vibrating cylinders at low Reynolds numbers," *Journal of Fluid Mechanics*, vol. 28, Pt. 3, 1967, pp. 501–512.

[14]Von Kármán, T., "Über Den Mechanismus Des Widerstandes, Den Ein Bewegter Körper In Einer Flüssigkeit Erfährt," *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen;* Mathematisch-Physikalische Klasse, 1912, pp. 547–556.

[15]Synge, J. L., "Mathematical Investigation of The Thrust Experienced By a Cylinder in a Current, The Motion Being Periodic," *Proceedings of the Royal Irish Academy*, Vol. 37, Ser. A, 1927, pp. 95–109.

It should be noted that Synge only references an early version of von Kármán's[14] paper which was published in 1911 in the Göttinger Nachrichten. The results given by von Kármán's in 1911 differ somewhat from the results he obtained from the more rigorous and more accurate formulations shown in his 1912 (Refs. 1 and 14) publications.